Answer All Questions

Q1. [**8 points**] If n^2+2 is odd, then $n+1$ is not odd.

$\forall n \in \mathbb{Z}$ n^2+2 is odd $\rightarrow$ $n+1$ is even

contrapositive

$\forall n \in \mathbb{Z}$ $n+1$ is odd $\rightarrow$ n^2+2 is even

~~n~~ let n is integer and $n+1$ is odd

$\cancel{n+1=2k+1}$

$\cancel{n=2k+2}$

$\cancel{n^2=4k^2+8k+4}$

$\cancel{n^2=4k^2+}$

$n+1=2k+1$

$n=2k+1-1$

8 $n=2k$

$n^2=4k^2$

$n^2+2=4k^2+2$

$n^2+2=2(2k^2+1)$

$n^2+2=2m$ where $m=2k^2+1 \in \mathbb{Z}$

$\therefore n^2+2$ is even Hence proved.

Q2. (a) [**4 points**] Determine all of the elements in each of the following sets.

2 (1) $\{1+(-1)^n \mid n \in \mathbb{Z}\} = \{2, 0\}$

(2) $\{x \in \mathbb{R} \mid x^2+4x+3=0\} = \{-1, -3\}$ $x = \dfrac{-4 \pm \sqrt{16-12}}{2} \Rightarrow \boxed{x=-1 \quad -3=x}$

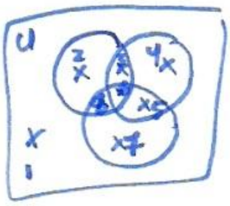
(b) [**6 points**] Let $A_n = \{n+1, n+2, n+3, \cdots\}$. Find $A_{n+1} = \{n+2, n+3, n+4, \cdots\}$

(1) $A_n \cup A_{n+1} = \{n+1, n+2, n+3, \ldots\} = A_n$

(2) $A_n \cap A_{n+1} = \{n+2, n+3, n+4, \ldots\} = A_{n+1}$

(3) $|\, A_n \cap \mathcal{P}(A_n) \,| = A_n \cap P(A_n) = \phi$ $P(A_n) = \{\phi, \{n+1\}, \cdots\}$

$\therefore |A_n \cap P(A_n)| = 0$ zero.



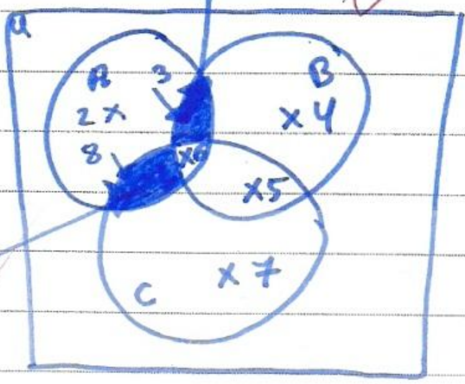
$A \cap B = \{3, 6\}$
$A \cap B - C = \{3\}$
$A \cap C = \{6, 8\}$
$A \cap C - B = \{8\}$

disjoints
$(A \cap B) - C \neq (A \cap C) - B$

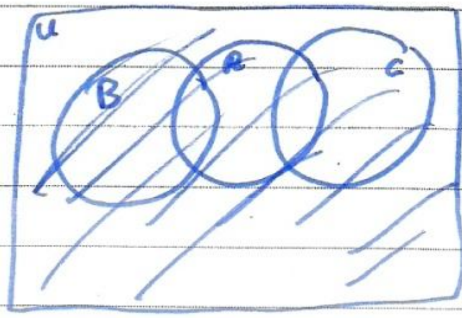
Q3. [**8 points**] For any sets A, B, and C, show that the sets $(A \cap B) - C$ and $(A \cap C) - B$ are disjoints.

same as up.

$A \cap B - C \Rightarrow \{3\}$



$A \cap C - B$
$\{8\}$



6

$\therefore (A \cap B) - C$ and $(A \cap C) - B$ are disjoints

How?

Q4. [**9 points**] Let $a, b, c \in \mathbb{Z}$ and $c \neq 0$. Prove that if c divides $(ca + b)$, the c divides b.

$\forall a, b, c \in \mathbb{Z}$ c divides $(ca + b) \longrightarrow c$ divides b

To conclude

Assume: c divides $(ca + b)$
$\therefore ca + b$ is divisible by c

~~Let c is an integer and $c \neq 0$ and~~
Let $ca + b$ an integer and divisible by c
$\therefore ca + b = ck$ $k \in \mathbb{Z}$

$b = ck - ca$
$b = c(k - a)$
$b = cm$ where $m \in \mathbb{Z}$ and $m = (k - a)$
$\therefore b$ is divisible by c
$\therefore c$ divides b
Hence proved.

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Q5. [**10 points**] Prove that

$$\frac{2}{3}+\frac{2}{3^2}+\cdots+\frac{2}{3^n}=1-\left(\frac{1}{3}\right)^n, \quad n\geq 1$$

Let $P(n): \frac{2}{3}+\frac{2}{3^2}+\cdots+\frac{2}{3^n}=1-\left(\frac{1}{3}\right)^n \quad n\geq 1$

Basis: show $P(1): \frac{2}{3^1}=1-\left(\frac{1}{3}\right)^{n=1} \Rightarrow \frac{2}{3}=\frac{2}{3}$

3 $\because$ R.H.S. = L.H.S $\therefore P(1)$ T

Assumption: Let $P(k)$: $\frac{2}{3}+\frac{2}{3^2}+\cdots+\frac{2}{3^k}=1-\left(\frac{1}{3}\right)^k \quad k\geq 1$

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Induction: show $P(k+1): \frac{2}{3}+\frac{2}{3^2}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}=1-\left(\frac{1}{3}\right)^{k+1}$ k≥

L.H.S $\Rightarrow \boxed{\frac{2}{3}+\frac{2}{3^2}+\cdots+\frac{2}{3^k}}+\frac{2}{3^k+1} \quad k\geq 1$

from assumption $\Downarrow$

$1-\left(\frac{1}{3}\right)^k+\frac{2}{3^{k+1}} \quad k\geq 1$

$1-\left(\frac{1}{3}\right)^k+\frac{2}{3^k\cdot 3} \Rightarrow 1-\frac{1^k}{3^k}+\frac{2}{3^k\cdot 3} \quad k\geq 1$

~~$1-\left[\frac{(1^k\cdot 3)+2}{3^k\cdot 3}\right] \Rightarrow$ $k\geq 1$~~

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$\frac{1\cdot 3^k-1^k}{3^k}+\frac{2}{3^k\cdot 3} \quad k\geq 1$

$\Rightarrow \frac{(3^k-1)3}{3^k\cdot 3}+\frac{2}{3^k\cdot 3} \quad k\geq 1$

$\frac{3^{k+1}-3+2}{3^{k+1}} \Rightarrow \frac{3^{k+1}-1}{3^{k+1}} \Rightarrow 1-\left(\frac{1}{3}\right)^{k+1} \quad k\geq 1$

$\therefore P(k+1)$

Hence proved.